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Mathematical Enrichment Jan 25th

1990 Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

[Problem from 1990 International Mathematical Olympiad in Beijing. Extremely difficult!]

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Number Theory

Problems about properties of integers or problems where we look for integer solutions

Find $\boxed{\text{all}}$ integer solutions of

$$x^2 + y^2 = z^2$$

Eg (3, 4, 5) . . . .

(5, 12, 13) . . . . .

⋮ infinitely many more families

Pythagorean Triples

$$x^3 + y^3 = z^3.$$

Prime numbers is an integer > 1 which has no divisors / factors other than 1 and itself.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$$

Much of number theory is about the mysteries of this sequence.

Eg: Does the sequence go on forever?

Show that there are 1000 consecutive integers all of which are composite.

$n > 1$ $\qquad 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots \cdot n = n!$

is div by $2, 3, 4, \ldots, n$.

$n! + 2$ is div. by 2

ie. $2 \mid n! + 2$

$3 \mid n! + 3$

$4 \mid n! + 4$

$\vdots$

All of the numbers

$$n!+2,\ n!+3,\ \ldots,\ n!+n \qquad (n-1 \text{ numbers})$$

are composite.

So take $\underline{n = 1001}$

Does the sequence of primes go on forever:

<u>Yes</u>.

<u>Proof:</u> (due to <u>Euclid</u> ca. 350 BC)

We will need the following <u>Theorem</u>:

Every integer greater than 1 is a product of primes. $\left(\begin{aligned} 30 &= 2\cdot 3\cdot 5 \\ 1500 &= 2^2\cdot 3\cdot 5^3 \end{aligned}\right)$

In particular, every integer >1 is divisible by a prime number.

Let $p_1, p_2, \ldots, p_n$ be any finite list of prime numbers.

It's enough to show that there is a prime not in this list.

This will follow if we can produce any integer >1 which is not divisible by any of the primes in our list (since it is divisible by <u>some</u> prime).

Here is such a number

$$N = p_1 \cdot p_2 \cdot p_3 \cdots \cdot p_n + 1$$

We get remainder 1 when we divide by p_1 or p_2 or . . . or p_n.

$\boxtimes$

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1. Let m be any odd number > 1

Show that there is a power of 2 which leaves remainder 1 when divided by m:

i.e. $2^n = mt + 1$ for some positive integers n, t.

2. Show that there is a sequence of positive integers $n_1 < n_2 < n_3 < \cdots\cdots$

Such that the numbers

$$2^{n_1} - 1,\ 2^{n_2} - 1,\ 2^{n_3} - 1,\ 2^{n_4} - 1, \cdots\cdots$$

are pairwise relatively prime:

Two numbers are relatively prime if they have no common prime divisor. (i.e. no common factor bigger than 1): 12, 35
7, 11

Can one find <u>14</u> consecutive positive integers each of which is divisible by at least one of $2, 3, 5, 7, 11$?